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A CHARACTERIZATION OF THE GROUP $U_3(4)$(1)

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Abstract. Let T be a Sylow 2-subgroup of the projective special unitary group $U_3(4)$, and let G be a finite group with Sylow 2-subgroups isomorphic to T. It is shown that if G is simple, then $G \cong U_3(4)$; if G has no proper normal subgroup of odd order or index, then $G \cong U_3(4)$ or T.

1. **Introduction.** We denote by $U_3(4)$ the projective special group of 3×3 unitary matrices with coefficients in the field of 4^2 elements. Let T be a Sylow 2-subgroup of $U_3(4)$. Our main result is

THEOREM 1. *Let G be a finite simple group whose Sylow 2-subgroups are isomorphic to T. Then $G\cong U_3(4)$.*

As a simple consequence we obtain

COROLLARY. *Let G be a finite group whose Sylow 2-subgroups are isomorphic to T. Suppose $O_{2'}(G)=G/O^{2'}(G)=1$. Then $G\cong U_3(4)$ or $G\cong T$.*

Theorem 1 can be applied to complete the proof of the following result of Janko and Thompson [11].

THEOREM. *Let G be a finite nonabelian simple group with Sylow 2-subgroup S. Assume that*

(a) $SCN_3(S)=\varnothing$,

(b) *$C_G(x)$ is solvable whenever x is an involution in S such that $|S:C_S(x)|\leqq 2$.*

Then G is isomorphic to A_7, M_{11}, $L_3(3)$, $U_3(3)$, $U_3(4)$, or $L_2(q)$ for q odd.

When the classification of finite simple groups with wreathed Sylow 2-subgroups is finished (see [1]), it will combine with results of MacWilliams [12], Alperin-Brauer-Gorenstein [1], Gorenstein-Walter [10], and with Theorem 1 to provide a classification of finite simple groups in which every elementary abelian 2-subgroup has rank at most 2. If no new groups turn up in the wreathed case, then the only such groups are $L_2(q)$, $L_3(q)$, $U_3(q)$ for q odd; A_7, M_{11}, and $U_3(4)$.

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We note the following well-known facts about T:

(i) $|T|=2^6$;

(ii) $Z(T)=T'=\Phi(T)=\Omega_1(T)=\mho^1(T)$ is a four-group.

With a little extra effort we can prove the following slight strengthening of Theorem 1:

THEOREM 2. *Let G be a finite simple group. Suppose a Sylow 2-subgroup T of G satisfies* (i) *and* (ii). *Then $G\cong U_3(4)$.*

The proof of Theorem 2 is patterned after the characterization of M_{12} by Brauer and Fong [7]. Namely, we compute the generalized decomposition numbers for the principal 2-block of a group G satisfying the hypotheses of Theorem 2, and then use group-order formulas to conclude that G has an ordinary rational character of degree 12. From the resulting bound on $|G|$ it follows easily that G has a strongly embedded subgroup and so is isomorphic to $U_3(4)$ by a theorem of Bender [2].

2. **2-local structure.** We begin the proof of Theorem 2. Let G be a finite simple group with a Sylow 2-subgroup T satisfying (i) and (ii). Let t be a fixed element of T of order 4, and let $z=t^2$.

LEMMA 1. (a) *G has one class of involutions and one class of elements of order* 4.

(b) *Elements of order* 4 *are rational but not strongly real.*

(c) $N_G(T)/O_{2'}(N_G(T))\cong T\langle\beta\rangle$, *where β is a fixed-point-free automorphism of T of order* 15.

(d) $C_G(z)/O_{2'}(C_G(z))\cong T\langle\beta^3\rangle$.

(e) $|C_G(t)/O_{2'}(C_G(t))|=2^4$.

Proof. By the Z^*-theorem [8], no involution in $Z(T)$ is weakly closed in T. Since $Z(T)=\Omega_1(T)$ contains just three involutions, they must all be fused in G. Hence G has one class of involutions. Moreover, by a result of Burnside, $K=N_G(T)/TC_G(T)$ contains an element α of order 3 acting fixed-point-free on T. In particular, all involutions in T have the same number (20) of square roots in T.

As $|T/\Phi(T)|=2^4$, $|K|\mid 3^2\cdot 5\cdot 7$. We claim $|K|=15$, which will prove (c). Suppose $x\in K$ has order 7. Then $|C_T(x)|=2^3$ and x centralizes $Z(T)$, so stabilizes each set of 20 square roots of elements of $Z(T)^{\#}$. Therefore $|C_T(x)|\geqq 3.6$, a contradiction. Hence $7\nmid |K|$. Suppose $9\mid |K|$; then K contains a Sylow 3-subgroup $\langle\alpha,\alpha_1\rangle$, where $\alpha_1^3=1$ and $|C_{T/\Phi(T)}(\alpha_1)|=4$. Then α_1 must centralize $Z(T)$, so $|C_T(\alpha_1)|=16$. Since α_1 commutes with α, it must fix the same number of square roots of each involution of T, and hence fixes 4 of each. Hence α_1 acts without fixed points on the remaining 16 square roots of each involution, which is absurd. Therefore $9\nmid |K|$.

Suppose $|K|=3$. Let $C=C_G(z)$. Obviously $Z(T)/\langle z\rangle$ is weakly closed in $T/\langle z\rangle$ with respect to $C/\langle z\rangle$, since $Z(T)=\Omega_1(T)$. By the Z^*-theorem, $Z(T)\leqq Z^*(C)$. Let $\bar{C}=C/O_{2'}(C)\cdot Z(T)$. As $|K|=3$, $\bar{C}$ has a Sylow 2-subgroup lying in the center of its normalizer; thus $\bar{C}$ has a normal 2-complement, so C does also. Moreover, we claim that $N_G(T)$ controls fusion of elements of T. This is clear for involutions. If

$t_1, t_2 \in T$ have order 4 and $t_1^g = t_2$ for some $g \in G$, then $t_1^2 = (t_2^2)^n$ for some $n \in N_G(T)$; hence $gn \in C_G(t_1^2)$ and $t_1^{gn} \in T$. As $C_G(t_1^2)$ has a normal 2-complement, t_1^{gn} is T-conjugate to t_1. Hence t_1^g is conjugate to t_1 in $N_G(T)$. Now by a theorem of Glauberman [9], G is a Suzuki group, which is absurd (e.g., $3 \mid |G|$). Therefore $|K| \neq 3$.

Hence $|K| = 15$, proving (c). We next prove (d). Let $C = C_G(z)$. As above, we have $Z(T) \leqq Z^*(C)$. Denote residues mod $Z(T)O_{2'}(C)$ by bars. Thus $|N_{\bar{C}}(\bar{T}) : C_{\bar{C}}(\bar{T})| = 5$. Let N be a minimal normal subgroup of $\bar{C}$. As $O_{2'}(\bar{C}) = 1$, $N \cap \bar{T} \neq 1$. But $N_{\bar{C}}(\bar{T})$ acts irreducibly on $\bar{T}$ so $\bar{T} \leqq N$. Now the main theorem of [14] implies that N is abelian, so $\bar{T} = N \lhd \bar{C}$. Thus $C = O_{2'}(C) \cdot N_C(T)$, which proves (d).

Next, β acts transitively on the elements of $(T/Z(T))^{\#}$. Hence the coset $tZ(T) = tT'$ contains representatives of all G-conjugacy classes of elements of order 4. Suppose that not all elements of $tZ(T)$ are fused in T, i.e. $|C_T(t)| > 2^4$. Then $|C_T(t)| = 2^5$ as $t \notin Z(T)$, and by applying β we conclude that $|C_T(x)| = 2^5$ if $x \in T - Z(T)$. This implies that T has $4+30$ conjugacy classes. Hence it has 16 linear characters and 18 ordinary characters of degree at least 2, so $|T| \geqq 16 + 4.18$, a contradiction. Therefore, all elements of $tZ(T)$ are fused, proving (a). Also, as $t^2 = z$, $C_G(t)/O_{2'}(C_G(t)) \simeq C_{T\langle\beta^3\rangle}(t)$ by (c); this equals $C_T(t)$, proving (e).

Finally, (b) is clear from the fact that T contains three involutions, hence no subgroup isomorphic to D_8.

3. **Generalized decomposition numbers of $B_0(G)$.** For any group H, we denote the principal 2-block of H by $B_0(H)$. We first determine the Cartan matrices C^z and C^t of $B_0(C_G(z))$ and $B_0(C_G(t))$. Since $C_G(t)$ has a normal 2-complement, $B_0(C_G(t))$ contains just one Brauer character and $C^t = (16)$ with respect to the basic set $\{1\}$. Let λ be a fixed linear character of $C_G(z)$ with kernel $T \cdot O_{2'}(C_G(z))$. Let μ be the restriction of λ to the elements of $C_G(z)$ of odd order.

LEMMA 2. *$(C^z)_{ij} = 4(3 + \delta_{ij})$ with respect to the basic set $\{1, \mu, \mu^2, \mu^4, \mu^3\}$.*

Proof. We may assume $O_{2'}(C_G(z)) = 1$; then since $Z(T) \leqq Z(C_G(z))$, it suffices to show that $C_{ij} = 3 + \delta_{ij}$ where C is the Cartan matrix of $B_0(T\langle\beta^3\rangle/Z(T))$ with respect to the μ^i's considered as Brauer characters modulo $Z(T)$. (See [5], [6].) One checks directly that each λ^i, hence each μ^i, is in the principal 2-block; since the μ^i are the only Brauer characters of $T\langle\beta^3\rangle/Z(T)$, all ordinary characters of this group lie in the principal 2-block. There are five linear characters, and three faithful ones, which equal $\sum_{i=0}^{4} \mu^i$ on elements of odd order. The lemma follows easily.

Let $1 = \chi_0, \chi_1, \ldots, \chi_m$ be the ordinary characters in $B_0(G)$. Then there exist generalized decomposition numbers d_j^t and ${}_id_j^z$, $1 \leqq i \leqq 5$, $0 \leqq j \leqq m$, such that

$$\chi_j(t\rho) = d_j^t \quad \text{and} \tag{3.1}$$
$$\chi_j(z\pi) = {}_1d_j^z + {}_2d_j^z\mu(\pi) + {}_3d_j^z\mu^2(\pi) + {}_4d_j^z\mu^4(\pi) + {}_5d_j^z\mu^3(\pi)$$

for all $\rho \in C_G(t)$ and $\pi \in C_G(z)$ of odd order. The ${}_id_j^z$ are automatically rational integers; since $\chi_j(t) = d_j^t$ and t is rational, the d_j^t are as well. We consider d^t and ${}_id^z$ to be columns of numbers indexed by $B_0(G)$, whose jth entries are d_j^t and ${}_id_j^z$

respectively. For any two columns A and B indexed by $B_0(G)$, put $(A, B)=\sum_{j=0}^{m} A_j\bar{B}_j$ (the bar denotes complex conjugation). By Lemma 2 and [3] we have

$$(3.2)\qquad \begin{aligned}(d^t, d^t) &= 16; \qquad (d^t, {}_id^z) = 0;\\ ({}_id^z, {}_jd^z) &= 4(3+\delta_{ij}) \quad \text{for } 1 \leqq i, j \leqq 5.\end{aligned}$$

The method of contribution [7] yields

$$(3.3)\qquad 4(d_j^t)^2+\sum_{i=1}^{5} ({}_id_j^z)^2+3\sum_{h<i} ({}_hd_j^z-{}_id_j^z)^2 < 64$$

for each j, $0\leqq j\leqq m$.

LEMMA 3. $\chi(z)\equiv\chi(t) \pmod 4$ *for any character* χ *of* G.

Proof. By Lemma 1, $|C_T(x)|=2^4$ for all $x\in T-Z(T)$. Since T has 2^4 linear characters all nonlinear characters of T vanish outside $Z(T)$. Let ψ be such a character not containing z in its kernel. Then $\psi(1)=4=-\psi(z)$ and so $(\chi|T, \psi)\in \boldsymbol{Z}$ implies $\chi(1)\equiv\chi(z) \pmod{16}$. Then summing χ on $C_T(t)$ yields $4\chi(z)+12\chi(t)\equiv 0 \pmod{16}$, proving the lemma.

Together with (3.1), Lemma 3 yields

$$(3.4)\qquad d_j^t \equiv \sum_{i=1}^{5} {}_id_j^z \pmod 4, \qquad 0 \leqq j \leqq m.$$

Let σ be a Galois automorphism of some splitting field for G, such that $\mu^\sigma=\mu^2$. Then for any χ_j in $B_0(G)$, χ_j^σ is also in $B_0(G)$ so there exists an index k, $0\leqq k\leqq m$, such that $\chi_j^\sigma=\chi_k$. From (3.1) we obtain $d_j^t=d_k^t$, ${}_1d_j^z={}_1d_k^z$, ${}_2d_j^z={}_5d_k^z$, ${}_3d_j^z={}_2d_k^z$, ${}_4d_j^z={}_3d_k^z$, ${}_5d_j^z={}_4d_k^z$. We refer to this fact as "Galois symmetry."

Now, using (3.2), (3.3), (3.4), and Galois symmetry, we shall show that the generalized decomposition numbers for $B_0(G)$ are one of the possibilities (A) through (V) listed in Table I, up to a sign change in each row and a permutation of rows. In each case, the jth row consists of d_j^t and ${}_id_j^z$, $i=1, 2, 3, 4, 5$. We denote by v_j the 5-tuple $({}_1d_j^z, {}_2d_j^z, {}_3d_j^z, {}_4d_j^z, {}_5d_j^z)$.

TABLE I

Possible sets of generalized decomposition numbers of $B_0(G)$

(A) Z_1 or Z_2

d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
1	1	0	0	0	0	y_1
1	1	0	0	0	0	y_2
1	1	2	2	2	2	y_3
2	2	2	2	2	2	

(B) Z_1 or Z_2

d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
1	1	1	1	1	1	y_1
1	1	1	1	1	1	y_2
1	1	2	2	2	2	y_3
2	2	1	1	1	1	y_4
0	0	1	1	1	1	y_5

(C)

	d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
				Z_1 or Z_2			
	1	1	0	0	0	0	y_1
	1	1	1	1	1	1	
	1	1	1	1	1	1	
	2	2	2	2	2	2	
	0	0	1	1	1	1	y_5
	0	0	1	1	1	1	y_6

(D)

	d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
				Z_1 or Z_2			
	1	1	0	0	0	0	
	1	1	0	0	0	0	
$4\times$	1	1	1	1	1	1	
	1	1	2	2	2	2	

(E)

				Z_1 or Z_2			
	1	1	0	0	0	0	
$6\times$	1	1	1	1	1	1	
	0	0	1	1	1	1	
	0	0	1	1	1	1	

(F)

				Z_3			
	1	1	1	1	1	1	
	1	1	1	1	1	1	
	1	1	0	0	0	0	
	0	0	1	1	1	1	
	0	0	1	1	1	1	

(G)

				Z_3			
	1	1	2	2	2	2	
	1	1	0	0	0	0	
	1	1	0	0	0	0	

(H)

				Z_4			
	0	2	2	1	2	1	
	0	2	1	2	1	2	
	-1	1	0	1	0	1	y_2
	-1	1	1	0	1	0	y_2
	-3	1	1	1	1	1	y_3
$3\times$	0	0	1	1	1	1	

(J)

				Z_4			
	-1	1	0	1	0	1	y_1
	-1	1	1	0	1	0	y_1
	-1	1	2	1	2	1	y_2
	-1	1	1	2	1	2	y_2
	2	2	1	1	1	1	y_3
	1	1	1	1	1	1	
	1	1	1	1	1	1	
	1	1	0	0	0	0	y_6
	0	0	1	1	1	1	y_7

(K)

				Z_4			
	-1	1	0	1	0	1	y_1
	-1	1	1	0	1	0	y_1
	-1	1	0	1	0	1	y_2
	-1	1	1	0	1	0	y_2
	-2	2	2	2	2	2	
	1	1	2	2	2	2	y_4
	1	1	0	0	0	0	y_5
	1	1	0	0	0	0	y_6

(L)

				Z_4			
	-1	1	0	1	0	1	y_1
	-1	1	1	0	1	0	y_1
	-1	1	0	1	0	1	y_2
	-1	1	1	0	1	0	y_2
	-2	2	2	2	2	2	y_3

(M)

	1	1	0	0	0	0	1
				Z_5			x_1
				Z_5			x_2
				Z_5			x_3
				Z_6			x_4
	-1	3	3	3	3	3	y_1

d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
1	1	1	1	1	1	
1	1	1	1	1	1	
1	1	0	0	0	0	y_6
0	0	1	1	1	1	y_7
0	0	1	1	1	1	y_8

d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
1	1	0	0	0	0	y_2
1	1	0	0	0	0	y_3
0	0	1	1	1	1	y_4

(N)	1	1	0	0	0	0	1
				Z_5			x_1
				Z_5			x_2
				Z_6			x_3
				Z_6			x_4
	-2	2	2	2	2	2	x_5
	1	1	0	0	0	0	x_6
	1	1	2	2	2	2	x_7
	1	1	0	0	0	0	x_8

(P)	1	1	0	0	0	0	1
				Z_5			x_1
				Z_5			x_2
				Z_6			x_3
				Z_6			x_4
	-2	2	2	2	2	2	x_5
	1	1	0	0	0	0	x_6
	1	1	1	1	1	1	x_7
	1	1	1	1	1	1	x_8
	0	0	1	1	1	1	x_9
	0	0	1	1	1	1	x_{10}

(Q)	1	1	0	0	0	0	1
				Z_5			x_1
				Z_6			x_2
				Z_6			x_3
				Z_6			x_4
	-3	1	1	1	1	1	x_5
	1	1	2	2	2	2	x_6
	1	1	0	0	0	0	x_7
	0	0	1	1	1	1	x_8

(R)	1	1	0	0	0	0	1
				Z_5			x_1
				Z_6			x_2
				Z_6			x_3
				Z_6			x_4
	-3	1	1	1	1	1	x_5
	1	1	1	1	1	1	x_6
	1	1	1	1	1	1	x_7
$3\times$	0	0	1	1	1	1	

(S)							
				Z_7			
				Z_6			x_2
				Z_6			x_3
	-2	0	1	0	1	0	x_4
	-2	0	0	1	0	1	x_4
	1	1	2	2	2	2	x_5
	1	1	1	1	1	1	x_6
	1	1	1	1	1	1	x_7
	0	0	1	1	1	1	x_8

(T)							
				Z_7			
				Z_6			x_2
				Z_5			x_3
	-1	1	2	1	2	1	x_4
	-1	1	1	2	1	2	x_4
	1	1	2	2	2	2	x_5
	2	2	1	1	1	1	x_6

(U)

d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
			Z_7			
			Z_6			x_2
			Z_5			x_3
-1	1	2	1	2	1	x_4
-1	1	1	2	1	2	x_4
2	2	2	2	2	2	x_5
1	1	0	0	0	0	x_6
0	0	1	1	1	1	x_7

(V)

	d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
				Z_7			
				Z_6			x_2
				Z_5			x_3
	-1	1	2	1	2	1	x_4
	-1	1	1	2	1	2	x_4
$4\times$	1	1	1	1	1	1	$x_5^{(i)}$ $(1 \leqq i \leqq 4)$
	1	1	0	0	0	0	x_6
	0	0	1	1	1	1	x_7

where

Z_1:	1	1	0	0	0	0	1
	-1	1	1	1	1	-1	x_1
	-1	1	1	1	-1	1	x_1
	-1	1	1	-1	1	1	x_1
	-1	1	-1	1	1	1	x_1
	-2	2	2	2	2	2	x_2
Z_2:	1	1	0	0	0	0	1
	-1	1	0	0	0	2	x_1
	-1	1	0	0	2	0	x_1
	-1	1	0	2	0	0	x_1
	-1	1	2	0	0	0	x_1
	-2	2	2	2	2	2	x_2
Z_3:	1	1	0	0	0	0	1
	1	1	2	0	1	1	x_1
	1	1	0	1	1	2	x_1
	1	1	1	1	2	0	x_1
	1	1	1	2	0	1	x_1
	-1	1	1	1	0	0	x_2
	-1	1	1	0	0	1	x_2
	-1	1	0	0	1	1	x_2
	-1	1	0	1	1	0	x_2
	-2	2	2	2	2	2	x_3

Z_4:	1	1	0	0	0	0	1
	1	1	2	1	0	1	x_1
	1	1	1	0	1	2	x_1
	1	1	0	1	2	1	x_1
	1	1	1	2	1	0	x_1
Z_5:	1	0	1	0	0	0	
	1	0	0	1	0	0	
	1	0	0	0	1	0	
	1	0	0	0	0	1	
Z_6:	0	1	1	1	1	0	
	0	1	1	1	0	1	
	0	1	1	0	1	1	
	0	1	0	1	1	1	
Z_7:	1	1	0	0	0	0	1
	-1	1	1	1	0	0	x_1
	-1	1	1	0	0	1	x_1
	-1	1	0	0	1	1	x_1
	-1	1	0	1	1	0	x_1

Define the following columns of rational integers indexed by $B_0(G)$: ${}_0A={}_1d^z-{}_2d^z$, ${}_1A={}_2d^z-{}_3d^z$, ${}_2A={}_3d^z-{}_4d^z$, ${}_3A={}_4d^z-{}_5d^z$, ${}_4A={}_5d^z-{}_2d^z$. Thus for any j, $\sum_{i=1}^4 {}_iA_j=0$, and by Galois symmetry there exists j' with ${}_0A_{j'}={}_0A_j+{}_1A_j$, ${}_iA_j={}_{i+1}A_{j'}$ $(i=1,2,3)$, ${}_4A_j={}_1A_{j'}$. From (3.2) we get $({}_iA, {}_iA)=8$ $(0\leqq i\leqq 4)$; $({}_1A, {}_4A)=({}_iA, {}_{i+1}A)=({}_0A, {}_4A)=-4$ $(0\leqq i\leqq 3)$; and $({}_0A, {}_2A)=({}_0A, {}_3A)=({}_1A, {}_3A)=({}_2A, {}_4A)=0$. We always take $\chi_0=1_G$; thus ${}_0A_0=1$, ${}_iA_0=0$ for $i>0$.

We consider first the case when some entry of some ${}_iA$, $i>0$, is ± 2. By Galois symmetry we may assume $i=1$, and since we are allowing permutations of rows and sign changes in each row, we may assume ${}_1A_1=2$. If ${}_3A_1=2$, then $({}_iA, {}_iA)=8$ and $\sum_{i=1}^4 {}_iA_1=0$ imply ${}_2A_1={}_4A_1=-2$. By Galois symmetry, we may assume ${}_1A_2={}_3A_2=-{}_2A_2=-{}_4A_2=-2$, contradicting $({}_1A, {}_3A)=0$. Therefore ${}_3A_1\neq 2$. Again by Galois symmetry, we may assume ${}_iA_j={}_jA_i$ for $1\leqq i, j\leqq 4$; ${}_1d_1^z={}_1d_2^z={}_1d_3^z={}_1d_4^z$; $d_1^t=d_2^t=d_3^t=d_4^t$; and ${}_0A_j={}_0A_{j-1}+{}_1A_{j-1}$ for $2\leqq j\leqq 4$. It follows easily from $({}_1A, {}_3A)=0$ that ${}_3A_1=0$. We consider the several possibilities for ${}_0A_1$ separately. Note that always $\chi_j(z)\neq 0$, for otherwise, $(d^t, d^t)=16$, $d_j^t=\chi_j(t)\equiv\chi_j(z) \pmod 4$ imply $\chi_j(t)=0$, whence χ_j has defect zero, contradicting $\chi_j\in B_0(G)$.

Case 1. ${}_0A_1>0$. Then ${}_0A_2={}_0A_1+{}_1A_1\geqq 2$, contradicting $({}_0A, {}_0A)=8$.

Case 2. ${}_0A_1=0$. By an argument like that in Case 1, we find ${}_1A_2={}_2A_1=-1$ or -2.

(a) Suppose ${}_1A_2=-2$. $({}_1A, {}_1A)=8$ yields ${}_1A_j=0$ for $j>2$. From $(d^t, d^t)=({}_1d^z, {}_1d^z)=16$, (3.3), and (3.4), we find $d_1^t=-1$ and $v_1=(1, 1, -1, 1, 1)$. For $j>4$ we have ${}_2d_j^z={}_3d_j^z={}_4d_j^z={}_5d_j^z$, so ${}_1d_j^z\equiv d_j^t \pmod 4$. Since $(d^t, {}_1d^z)=0$, we may assume $d_5^t=-2$, ${}_1d_5^z=2$. Then clearly $d_j^t={}_1d_j^z$ for $j>5$. For $i>1$, $({}_id^z, {}_1d^z-d^t)=12$ yields ${}_id_5^z=2$. It now follows easily that we have one of the cases (A)–(E) of Table I, with Z_1's.

(b) Suppose ${}_1A_2=-1$. Then ${}_1A_3=0$ implies ${}_1A_4=-1$. It follows that $\sum_{j=0}^4 {}_0A_j^2=7$; since $({}_0A, {}_0A)=8$, we may assume ${}_0A_5=1$, ${}_0A_j=0$ for $j>5$. The conditions on $({}_0A, {}_iA)$ imply ${}_1A_5={}_2A_5=-{}_3A_5=-{}_4A_5=-1$. By Galois symmetry there exist at least four $j>4$ with ${}_1A_j\neq 0$, contradicting $({}_1A, {}_1A)=8$.

Case 3. ${}_0A_1=-1$. Suppose first that ${}_2A_1$ or ${}_4A_1$ is ± 2. As $({}_1A, {}_1A)=8$, it must be -2. If ${}_4A_1=-2$ we replace the first row by the fourth with a sign change and so may assume ${}_2A_1=-2$; then ${}_3A_1={}_4A_1=0$. As in Case 2(a), we easily conclude that we may assume $d_1^t=-1$ and $v_1=(1, 0, 2, 0, 0)$.

As in Case 2(a) we may assume ${}_jd_5^z=-d_5^t=2$, and we get (A)–(E) in Table I, with Z_2's.

Now suppose $|{}_iA_1|\leqq 1$, $2\leqq i\leqq 4$. It follows that ${}_2A_1={}_4A_1=-1$. As $\chi_1(z)\neq 0$, we may assume by (3.3), (3.4), that $d_1^t=1$ and $v_1=(1, 2, 0, 1, 1)$ or $(1, 0, 2, 1, 1)$.

The arguments in both these cases are the same so we consider only the first. We have $\sum_{j=1}^4 {}_1A_j\,{}_3A_j=2$. Since $({}_1A, {}_3A)=0$ and $|{}_jA_k|\leqq 1$ for $j>0$, $k>4$, we may assume ${}_1A_5=-{}_3A_5=1$. By Galois symmetry, χ_5 has at least four algebraic conjugates under σ, and it follows easily from $({}_iA, {}_iA)=8$ that ${}_2A_5={}_4A_5=0$. We may assume $\chi_5^{\sigma^i}=\chi_{5+i}$, $0\leqq i\leqq 3$. As $({}_0A, {}_0A)=8$, ${}_0A_5=0$ or -1. By replacing the fifth row with the seventh with a sign change if necessary, we may assume ${}_0A_5=0$. Then we may assume $d_5^t=-1$, and $v_5=(1, 1, 0, 0, 1)$. As in 2(a) we may assume $d_9^t=-{}_1d_9^z=-2$; $({}_jd^z, {}_1d^z-d^t)=12$ yields ${}_jd_9^z=2$ for $2\leqq j\leqq 5$. We then clearly get (F) or (G) in Table I.

Case 4. ${}_0A_1=-2$. If ${}_2A_1=-2$, then ${}_0A_2=0$ and ${}_1A_2=-2$, and Case 3 applies. Similarly, if ${}_4A_1=-2$, then ${}_0A_4=0$ and ${}_1A_4=-2$ and Case 3 applies again. As $\sum_{i=1}^4 {}_iA_1=0$, we may assume ${}_2A_1={}_4A_1=-1$; an argument like that in Case 2(b) gives a contradiction.

Now we may assume that $|{}_iA_j| \leqq 1$ for $i \geqq 1$ and all j.

Case 5. ${}_1A_1 = {}_2A_1 = -{}_3A_1 = -{}_4A_1 = 1$. By Galois symmetry, we may assume ${}_1A_2 = {}_4A_2 = {}_3A_3 = {}_4A_3 = {}_2A_4 = {}_3A_4 = 1$ and other ${}_iA_j$, $2 \leqq i \leqq 4$, $1 \leqq j \leqq 4$, are -1. The conditions on the inner products $({}_iA, {}_jA)$ imply that we may assume ${}_1A_5 = {}_2A_6 = {}_1A_7 = {}_2A_8 = {}_3A_5 = {}_4A_6 = {}_3A_7 = {}_4A_8 = 1$ and other ${}_iA_j$, $5 \leqq j \leqq 8$, $1 \leqq i \leqq 4$, are -1. From $({}_0A, {}_0A) = 8$ we may assume ${}_0A_1 = -{}_0A_3 = {}_0A_5 = {}_0A_7 = -1$, ${}_0A_2 = {}_0A_4 = {}_0A_6 = {}_0A_8 = 0$. We then have $d_j^t \equiv {}_1d_j^z \pmod 4$ for $j > 8$. (3.3) and $\chi_1(z) \neq 0$ imply that we may assume $d_1^t = 1$, $v_1 = (1, 2, 1, 0, 1)$, by replacing the first row with the third with a sign change, if necessary. By Galois symmetry, $({}_2d^z, {}_2d^z) = 16$, and (3.3), we have $v_5 = (1, 2, 1, 2, 1)$, $(0, 1, 0, 1, 0)$, $(1, 0, 1, 0, 1)$ or $(2, 1, 2, 1, 2)$, and similarly for v_7.

(a) If $v_5 = (2, 1, 2, 1, 2)$, then $d_5^t = 0$; (3.2) implies $v_7 = (1, 0, 1, 0, 1)$ and $d_7^t = -1$. Then $(d^t, {}_1d^z) = 0$ implies we may assume $d_9^t = -3$, ${}_1d_9^z = 1$. $({}_jd^z, d^t) = 0$ yields ${}_jd_9^z = 1$ for $2 \leqq j \leqq 5$ and we have (H) in Table I.

(b) If $v_5 = (0, 1, 0, 1, 0)$, then $d_5^t = \pm 2$. If $d_5^t = 2$, then the Schwarz inequality on the columns $(d_j^t)_{j>6}$ and $({}_2d_j^z)_{j>6}$ yields $6 \leqq 27^{1/2}$, a contradiction. So $d_5^t = -2$. Now $(d^t, d^t) = 16$ implies $v_7 = (1, 0, 1, 0, 1)$ or $(1, 2, 1, 2, 1)$; $(d^t, {}_1d^z) = 0$ implies we may assume $d_9^t = -1$, ${}_1d_9^z = 3$, and so $({}_1d^z, {}_2d^z) \not\equiv 0 \pmod 3$, a contradiction.

(c) We may now assume that v_5 and v_7 are either $(1, 0, 1, 0, 1)$ or $(1, 2, 1, 2, 1)$. If both are $(1, 2, 1, 2, 1)$, then $\sum_{j=0}^{8} ({}_2d_j^z)^2 = 16$ and $\sum_{j=0}^{8} {}_1d_j^z\, {}_2d_j^z = 10$, a contradiction. Hence we may assume $v_5 = (1, 0, 1, 0, 1)$, $d_5^t = -1$. As $(d^t, {}_1d^z) = 0$, we may assume $d_9^t = -{}_1d_9^z = -2$, $d_j^t = {}_1d_j^z = 1$, $10 \leqq j \leqq 12$; we easily get (J), (K), or (L) in Table I. This disposes of Case 5.

Since $\sum_{i=1}^{4} {}_iA_j = 0$ for all j, we may assume that for each j, $({}_iA_j)_{i=1}^{4}$ is some cyclic permutation of $(1, 0, -1, 0)$; $(1, -1, 1, -1)$; $(1, -1, 0, 0)$; or $(0, 0, 0, 0)$, possibly with a sign change. (3.2), (3.3), (3.4), Galois symmetry, $\chi_j(z) \neq 0$, and the conditions on $({}_iA, {}_jA)$ yield the possibilities for v_j shown in Table II.

Case 6. No $({}_iA_j)_{i=1}^{4}$ is $(1, 0, -1, 0)$. Then since $({}_1A, {}_3A) = 0$, no $({}_iA_j)_{i=1}^{4}$ is $(1, -1, 1, -1)$. Hence we may assume $({}_iA_{4n+1})_{i=1}^{4} = (1, -1, 0, 0)$ and $\chi_{4n+k} = \chi_{4n+1}^{\sigma^{k-1}}$, $0 \leqq n \leqq 3$, $1 \leqq k \leqq 4$. If ${}_0A_1 = -1$, we get ${}_0A_3 = {}_0A_4 = -1$. Since ${}_0A_0 = 1$ and $({}_0A, {}_0A) = 8$, we may assume ${}_0A_1 = {}_0A_5 = {}_0A_9 = 0$. We have $d_j^t \equiv {}_1d_j^z \pmod 4$ for $j > 16$.

(a) ${}_0A_{13} = 0$. Then v_1, v_5, v_9, and v_{13} are each either $(0, 0, -1, 0, 0)$ or $(1, 1, 0, 1, 1)$. Correspondingly, d_1^t, d_5^t, d_9^t, and d_{13}^t are either -1 or 0. Since $({}_1d^z, {}_1d^z) = (d^t, d^t) = 16$, we cannot have $v_1 = v_5 = v_9 = v_{13}$. Depending on whether one, two, or three of v_1, v_5, v_9, and v_{13} are $(1, 1, 0, 1, 1)$, we get (by permuting rows and changing signs) cases (M); (N) or (P); (Q) or (R) in Table I.

(b) ${}_0A_{13} = -1$. If $v_{13} = (1, 2, 1, 2, 2)$, then, by $({}_2d^z, {}_2d^z) = 16$, $v_1 = v_5 = v_9 = (0, 0, -1, 0, 0)$, against $({}_1d^z, {}_2d^z) = 12$. So $v_{13} = (0, 1, 0, 1, 1)$. Thus $d_{13}^t = -1$, and as $(d^t, d^t) = 16$, we may assume $v_9 = (1, 1, 0, 1, 1)$. Suppose that k of v_5 and v_7 are $(0, 0, -1, 0, 0)$. By $(d^t, {}_1d^z) = 0$, we may assume ${}_1d_{17}^z = k + 1$ and $d_{17}^t = k - 3$ ($k = 0, 1$, or 2). From $({}_2d^z, {}_1d^z - d^t) = 12$, we get ${}_2d_{17}^z = k$. The Schwarz inequality on $({}_1d_j^z)_{j \geqq 17}$ and $({}_2d_j^z)_{j \geqq 17}$ yields $(3 + 2k - k^2)^2 \leqq (4 + 2k - k^2)(2 + 2k - k^2)$ which is impossible for $0 \leqq k \leqq 2$.

TABLE II

Possible v_j for given $({}_iA_j)_{i=1}^4$

$({}_iA_j)_{i=1}^4$	${}_0A_j$	Possible v_j (up to sign change and Galois conjugacy)
$(1, 0, -1, 0)$	0	$(1, 1, 0, 0, 1)$
		$(1, 1, 2, 2, 1)$
$(1, -1, 1, -1)$	0	$(2, 2, 1, 2, 1)$
		$(1, 1, 0, 1, 0)$
		$(0, 0, 1, 0, 1)$
		$(1, 1, 2, 1, 2)$
$(1, -1, 1, -1)$	1	$(2, 1, 0, 1, 0)$
		$(1, 0, -1, 0, -1)$
		$(0, 1, 2, 1, 2)$
$(1, -1, 0, 0)$	0	$(1, 1, 0, 1, 1)$
		$(0, 0, -1, 0, 0)$
$(1, -1, 0, 0)$	-1	$(1, 2, 1, 2, 2)$
		$(0, 1, 0, 1, 1)$

Case 7. $({}_iA_j)_{i=1}^4=(1, 0, -1, 0)$ for two distinct values of j, say $j=1$ and $j=5$. Then ${}_1d_j^z=-d_j^t=\pm 1$ for $1\leqq j\leqq 8$, by Table II. As $({}_1d^z, d^t)=0$, we get ${}_1d_j^z=d_j^t$ for $j>8$. Since $({}_1A, {}_3A)=0$, we may assume $({}_iA_9)_{i=1}^4=(1, -1, 1, -1)$; then Table II and (3.4) yield ${}_1d_9^z\neq d_9^t$, a contradiction.

Case 8. We may now assume $({}_iA_1)_{i=1}^4=(1, 0, -1, 0)$, $({}_iA_5)_{i=1}^4=(1, -1, 1, -1)$, $({}_iA_7)_{i=1}^4=({}_iA_{11})_{i=1}^4=(1, -1, 0, 0)$. Since $({}_0A, {}_0A)=8$, ${}_0A_1={}_0A_5={}_0A_7=0$; ${}_0A_{11}=-1$ or 0. From Table II, $\sum_{j=7}^{10}(d_j^t-{}_2d_j^z)^2=3$, $\sum_{j=11}^{15}(d_j^t-{}_2d_j^z)^2\geqq 3$. If $v_1=(1, 1, 2, 2, 1)$, then we get $(d^t-{}_2d^z, d^t-{}_2d^z)>32$, a contradiction. Therefore, $v_1=(1, 1, 0, 0, 1)$, and $d_1^t=-1$. Suppose ${}_0A_{11}=-1$. Then ${}_0A_j=0$ for $j>14$; $({}_2d^z, {}_2d^z)=16$ implies $v_{11}=(0, 1, 0, 1, 1)$, so $d_{11}^t=-1$. Now $(d^t-{}_1d^z, d^t-{}_1d^z)=32$ implies $d_j^t={}_1d_j^z$ for $j>14$. Hence $4+\sum_{j=5}^{10}(d_j^t)^2=\sum_{j=5}^{10}({}_1d_j^z)^2=\sum_{j=5}^{10}({}_2d_j^z)^2$, as $(d^t, d^t)=({}_1d^z, {}_1d^z)=({}_2d^z, {}_2d^z)$. None of the possibilities for v_j, $5\leqq j\leqq 10$, listed in Table II satisfy these equations.

Therefore ${}_0A_{11}=0$. As above, ${}_1d_j^z=d_j^t$ for $j>14$. If $v_7=v_{11}=(0, 0, -1, 0, 0)$, then $(d^t, d^t)=({}_1d^z, {}_1d^z)=16$ implies $v_5=(2, 1, 2, 1, 2)$, so $({}_2d^z, {}_1d^z-d^t)=8$, a contradiction. Therefore we may assume $v_{11}=(1, 1, 0, 1, 1)$. If $v_7=(1, 1, 0, 1, 1)$, then $(d^t, d^t)=({}_1d^z, {}_1d^z)$ and $({}_2d^z, {}_1d^z-d^t)=12$ imply $v_5=(0, 1, 0, 1, 0)$ and $d_5^t=-2$. This yields (S) in Table I. Finally, if $v_7=(0, 0, -1, 0, 0)$, then $({}_2d^z, {}_1d^z-d^t)=12$ implies $v_7=(1, 2, 1, 2, 1)$, and we easily get (T), (U), or (V).

4. **Character degrees.** We show in this section that either case (U) or (V) in Table I holds, that G has a rational character of degree 12, and

$$|G| = 195|C_G(z)|^3/|C_G(Z(T))|^2. \tag{4.1}$$

Let d be one of the columns d^t or ${}_id^z$; let $x=t$ or z, respectively. Let $\tilde{G}=C_G^*(x)$ and let $\tilde{d}$ be the corresponding column of generalized decomposition numbers for $B_0(\tilde{G})$ (with respect to the basic set $\{1\}$ if $x=t$, $\{1, \mu, \mu^2, \mu^4, \mu^3\}$ if $x=z$). Let $\tilde{\chi}_0, \tilde{\chi}_1, \ldots, \tilde{\chi}_n$ be the ordinary characters in $B_0(\tilde{G})$ and define $h_j=\sum_\alpha \tilde{\chi}_j(z_\alpha)/|C_{\tilde{G}}(z_\alpha)|$, where z_α runs over $\tilde{G}$-conjugacy classes of involutions. Then by a result of Brauer [4],

$$|G| \sum_{j=0}^{m} \chi_j(z)^2 d_j/\chi_j(1) = |\tilde{G}|\ |C_G(z)|^2 \sum_{j=0}^{n} h_j^2 \tilde{d}_j/\tilde{\chi}_j(1).$$

We denote the left and right sides of this equation by $\mathrm{L}(d)$ and $\mathrm{R}(d)$, respectively. If A is any column indexed by $B_0(G)$ which is a linear combination of d^t and the ${}_id^z$, $\mathrm{L}(A)$ is defined as the corresponding linear combination of $\mathrm{L}(d^t)$ and the $\mathrm{L}({}_id^z)$.

LEMMA 4. (a) $\mathrm{L}(d^t)=0$.

(b) $\mathrm{L}({}_1d^z-{}_2d^z)=0$.

(c) $\mathrm{L}({}_1d^z)=128|C_G(z)|^3/|C_G(Z(T))|^2$.

Proof. Lemma 1(b) and a result of Brauer [4] imply (a). By Lemma 1, $|C_T(x)|=2^4$ for all elements x of T of order 4, so T has 16 linear characters and three irreducible characters ψ, ψ^β, and ψ^{β^2} of degree 4 vanishing off $Z(T)$. Choose notation so that $\ker \psi=\langle z\rangle$.

Let $\bar{C}=C_G(z)/O_{2'}(C_G(z))$. As argued in Lemma 2, all characters of $\bar{C}$ lie in $B_0(\bar{C})$. Let $\bar{T}$ be the image of T in $\bar{C}$. Thus $\bar{T}\cong T$.

The characters 1_T, ψ, ψ^β, ψ^{β^2} are all invariant in $\bar{C}$ and hence extend in five ways each to $\bar{C}$. Since $\exp \bar{C}=20$ and ψ is rational, it is easily seen that at least one extension $\tilde{\psi}$ of ψ is rational, whence $\tilde{\psi}(zf)=-1$ for all $f\in \bar{C}$ of order 5. The extensions of ψ are then $\tilde{\psi}\lambda^i$, $0\leqq i\leqq 4$. We have $\tilde{\psi}\lambda^i(zf)=\sum_{j\neq i}\lambda^j(f)$ for all $f\in\bar{C}$ of odd order. Hence the generalized decomposition numbers at z for the characters $\tilde{\psi}\lambda^i$ are the cyclic permutations of (0, 1, 1, 1, 1). Similarly, those for $\tilde{\psi}^\beta\lambda^i$ and $\tilde{\psi}^{\beta^2}\lambda^i$ are the cyclic permutations of (0, -1, -1, -1, -1). Finally, the fifteen linear nonprincipal characters of $\bar{T}$ form three orbits under the action of $\bar{C}$ and so by induction to $\bar{C}$ yield three irreducible characters of $\bar{C}$ of degree 5 vanishing off $\bar{T}$ and with $\bar{z}$ in their kernels. Thus the generalized decomposition numbers for each of these characters at $\bar{z}$ are (1, 1, 1, 1, 1). Now expand $\mathrm{R}({}_1d^z-{}_2d^z)$ from its definition. Apart from a constant factor, there is a sum of terms indexed by $B_0(\bar{C})$. It is clear that the only nonzero terms arise from 1 and λ; $\tilde{\psi}$ and $\tilde{\psi}\lambda$; $\tilde{\psi}^\beta$ and $\tilde{\psi}^\beta\lambda$; $\tilde{\psi}^{\beta^2}$ and $\tilde{\psi}^{\beta^2}\lambda$; and these cancel in pairs, proving (b). Put $c(z)=|C_G(z)|$, $c(Z(T))=|C_G(Z(T))|$. The $\bar{C}$-classes of involutions are represented by $\bar{z}$, $\bar{y}$, and $\bar{y}\bar{z}$ where $Z(T)=\langle y, z\rangle$. We find

$$\begin{aligned}\mathrm{R}({}_1d^z) &= c(z)^3\Bigg[\left(\frac{1}{c(z)}+\frac{2}{c(Z(T))}\right)^2+\left(\frac{4}{4}\right)\left(\frac{4}{c(z)}-\frac{8}{c(Z(T))}\right)^2 \\ &\qquad -\left(\frac{4}{4}\right)\left(-\frac{4}{c(z)}\right)^2-\left(\frac{4}{4}\right)\left(-\frac{4}{c(z)}\right)^2+\left(\frac{3}{5}\right)\left(\frac{5}{c(z)}+\frac{10}{c(Z(T))}\right)^2\Bigg] \\ &= 128c(z)^3/c(Z(T))^2,\end{aligned}$$

proving (c).

LEMMA 5. (a) $\chi_j(1) \geqq 12$ *if* $0 < j \leqq m$.

(b) $\chi_j(1) + 3\sum_{i=1}^{5} {}_id_j^z + 60d_j^t$ *is a nonnegative integral multiple of* 64.

(c) $\sum_{j=0}^{m} \chi_j(1)d_j^t = \sum_{j=0}^{m} \chi_j(1)_id_j^z = 0$ *for each i.*

Proof. (a) Since $\chi_j|T$ is faithful and $(\chi_j|T, \psi) = (\chi_j^\beta|T, \psi^\beta) = (\chi_j|T, \psi^\beta) = (\chi_j|T, \psi^{\beta^2})$, $\chi_j|T$ must contain $\psi + \psi^\beta + \psi^{\beta^2}$. (b) simply restates that $(\chi|T, 1_T)$ is a nonnegative integer; (c) is due to Brauer [3].

We shall use also the following consequence of a theorem of Schur [13]:

(*) If $\chi_j(1) = e > 5$ and $Q(\chi_j) \subseteq Q(\lambda)$, then no prime divisor of $|G|$ exceeds $e+1$.

The jth row of the column $\pm\chi(1)$ in Table I is defined as $\pm\chi_j(1)$, according as the jth row of generalized decomposition numbers for G is $\pm$ the jth row in Table I. We now eliminate (A)–(T) case by case.

(A) $\mathrm{L}({}_1d^z - {}_2d^z) = (\pm\chi(1), {}_1d^z - {}_2d^z) = 0$ yield

(A1) $1 + (18/x_1) + (1/y_1) + (1/y_2) - (81/y_3) = 0$,

(A2) $1 + 2x_1 + y_1 + y_2 - y_3 = 0$.

By Lemma 5(b), $x_1 \equiv 51$, $y_1 \equiv 1$, $y_2 \equiv 1$, $y_3 \equiv 41 \pmod{64}$. If $x_1 > 0$ or $x_1 < -77$, (A1) implies $y_3 = 41$, whence $(18/51) + (1/65) + (1/65) \geqq (18/x_1) + (1/y_1) + (1/y_2) = 40/41$, a contradiction. Therefore $-77 \leqq x_1 < 0$. If $x_1 = -13$, (A1) implies $y_3 < 0$; it is clear from Table I that $Q(\chi_1) \subseteq Q(\lambda)$, so (*) implies $y_3 < -343$, so $(2/65) + (81/343) \geqq (1/y_1) + (1/y_2) - (81/y_3) = 5/13$, a contradiction. Hence $x_1 = -77$. $|(1/y_1) + (1/y_2)| \leqq 2/63$ implies $|(59/77) - (81/y_3)| \leqq 2/63$, so $y_3 = 105$. Then $(1/y_1) + (1/y_2) = 2/385$, and (A2) gives $y_1 + y_2 = 258$. These equations have no solution, so (A) is impossible. (D) and (G) yield the same equations as (A) and so are also impossible.

(B) From $(\pm\chi(1), {}_1d^z - d^t) = 0$ we get $x_2 = -2x_1$. Then $\mathrm{L}({}_1d^z - d^t) > 0$ implies $x_1 < 0$. $\mathrm{L}({}_1d^z - {}_2d^z) = \mathrm{L}({}_1d^z - {}_2d^z + d^t) = (\pm\chi(1), {}_1d^z - {}_2d^z) = (\pm\chi(1), d^t) = 0$ yield

(B1) $1 + (18/x_1) - (81/y_3) + (36/y_4) - (16/y_5) = 0$,

(B2) $2 + (82/x_1) + (25/y_1) + (25/y_2) + (108/y_4) - (16/y_5) = 0$,

(B3) $1 + 2x_1 - y_3 + y_4 - y_5 = 0$,

(B4) $1 + y_1 + y_2 + y_3 + 2y_4 = 0$.

From Lemma 5(b), $x_1 \equiv 51$, $y_1 \equiv 53$, $y_2 \equiv 53$, $y_3 \equiv 41$, $y_4 \equiv 54$, $y_5 \equiv 52 \pmod{64}$; $x_2 \geqq 90$, and if $y_4 < 0$, then $y_4 \leqq -138$. Since $x_2 = -2x_1$, we get $x_1 \leqq -77$. Adding (B3) and (B4), we find that we cannot have $y_1, y_2, y_4 < 0$, $y_5 > 0$ at the same time. Suppose $x_1 < -77$. Then by (B2), we get $y_4 = -138$, $x_1 = -141$, and we may assume $y_1 = -75$. If $y_2 < 0$, then (B4) implies $y_3 \geqq 425$, and subtracting (B1) from (B2) yields $81/425 > (64/141) + (25/75) + (72/138) - 1$, a contradiction. So $y_2 > 0$; by (B2), $y_5 = 52$; (B3) yields $y_3 = 471 = 3 \cdot 157$, violating (*) applied to a character of degree 141. Therefore, $x_1 = -77$. If $y_4 > 0$, (B2) implies $y_1 = y_2 = -75$, $y_5 = 52$; (B3) and (B4) give $y_4 = 118$, violating (*) as $y_5 = 52$. Therefore $y_4 < 0$, and (B3) implies either y_3 or $y_5 < 0$. If $y_3 < 0$, (B1) implies $1 < (18/77) + (36/138) + (16/52)$, a contradiction. Therefore $y_5 < 0$, and (B1) implies $y_3 = 41$ or 105. If $y_3 = 41$, (B1) implies $y_5 = -12$, violating (*); therefore $y_3 = 105$. Subtracting (B1) from (B2) yields $(25/y_1) + (25/y_2)$

$<-(81/105)-(13/77)+(72/138)$. By (B4) we may assume $y_1>0$, so $25/y_2<-1/3$, $-75<y_2<0$, which is impossible.

(C) $L({}_1d^z-{}_2d^z)=0$ yields $1+(18/x_1)+(1/y_1)-(16/y_5)-(16/y_6)=0$. From Lemma 5(b), $x_1\equiv 51$, $y_1\equiv 1$, $y_5\equiv 52$, $y_6\equiv 52 \pmod{64}$. As in (B) we get $x_1<0$. If $x_1<-13$, then the above equation gives $(1/63)+(16/52)\cdot 2\geqq 59/77$, a contradiction. Thus $x_1=-13$. If $y_5<-12$, (*) implies $y_5<-140$; similarly for y_6. If both are <-12, we get $-1/y_1>(59/77)-(32/140)$, against $|y_1|\geqq 63$. Thus we may assume $y_5=-12$; then $(1/y_1)-(16/y_6)=37/39$, violating $y_6\equiv 52 \pmod{64}$. Cases (E) and (F) yield similar contradictions.

(H) $L(d^t)=(\pm\chi(1), d^t)=0$ yield

(H1) $1+(100/x_1)-(18/y_2)-(75/y_3)=0$,

(H2) $1+4x_1-2y_2-3y_3=0$.

We have $x_1\equiv 53$, $y_2\equiv 51$, $y_3\equiv 37 \pmod{64}$, and if $y_3>0$, then $y_3\geqq 165$. It follows easily from (H1) that $x_1<0$. By (H2) either $y_2<0$ or $y_3<0$. Therefore $100/x_1>1-(75/165)$, and $x_1=-75$ or -139. In either case (H1) and (H2) yield a quadratic equation for y_3 which has no integral solutions, a contradiction.

(J) $L({}_1d^z-{}_2d^z)=(\pm\chi(1), {}_1d^z-{}_2d^z)=(\pm\chi(1), d^t-{}_1d^z)=0$ yield

(J1) $1+(9/y_1)-(49/y_2)+(36/y_3)+(1/y_6)-(16/y_7)=0$,

(J2) $1+y_1-y_2+y_3+y_6-y_7=0$,

(J3) $y_1+y_2+y_3=0$.

$y_1\equiv 51$, $y_2\equiv 39$, $y_3\equiv 38$, $y_6\equiv 1$, $y_7\equiv 52 \pmod{64}$. By Lemma 4,

$$L(-3{}_1d^z+4{}_2d^z-d^t) > 0,$$

and this easily yields $y_2=39$ or 103. However, if $y_2=103$, then (*) implies $|y_j|\geqq 102$, $j=1, 3, 6$, and 7; the congruences and (J1) yield a contradiction. Therefore $y_2=39$. Suppose $y_1\geqq -13$. By (J3), $(9/y_1)+(36/y_3)<0$, and (J1) implies $y_7=-12$, $y_1=-13$, $y_3=-26$, $|y_6|\leqq 2$, a contradiction. Therefore $y_1<-13$, $y_3>0$, and $(36/y_3)+(9/y_1)>0$. From (J1), $y_7\neq -12$, otherwise $|y_6|<1$. Now if $y_1=-77$, then (J1) yields $0<y_7<40$, which is impossible. Applying (*) to a character of degree 39, we find $y_1\leqq -333$. The function $(9/y_1)+(36/-y_1-39)$ is increasing for $y_1<0$, so (J1) implies $y_7>-112$, $y_7<0$. Therefore $y_7=-76$. Now (J2) and (J3) imply $y_6=1$, a contradiction.

(K) $L({}_1d^z-{}_2d^z)=(\pm\chi(1), {}_1d^z-{}_2d^z)=0$ yield

(K1) $1+(9/y_1)+(9/y_2)-(81/y_4)+(1/y_5)+(1/y_6)=0$,

(K2) $1+y_1+y_2-y_4+y_5+y_6=0$.

$y_1\equiv 51$, $y_2\equiv 51$, $y_4\equiv 41$, $y_5\equiv 1$, $y_6\equiv 1 \pmod{64}$. If $y_1=y_2$, we can argue as in (A) to a contradiction. So we may assume $y_1\neq y_2$. If neither is -13, (K1) implies $y_4<105$, $y_4>0$, so $y_4=41$; thus from (K1), $(40/41)+(9/y_1)+(9/y_2)+(1/y_5)+(1/y_6)=0$, which is impossible. We may thus assume $y_1=-13$. As $y_2\neq y_1$, $Q(\chi)\subseteq Q(\lambda)$ where χ is a character of degree 13, and (*) applies. Now (K1) yields $y_4<540$, $y_4>0$. If $y_4\leqq 169$, then (K1) implies $(-9/y_2)+(1/y_5)+(1/y_6)\geqq 29/169$, so $y_2=51$, violating (*). It follows from (*) that $y_4=297$. Suppose $y_5<-63$. Then (*) implies $y_5<-500$; similarly for y_6. It follows easily from (K1) that we may assume $y_5=-63$. (K1) and (K2) yield $(9/y_2)+(1/y_6)=-172/9009$, $y_2+y_6=372$. Therefore $y_6>0$, $y_2<0$; (*) implies $y_2=-77$, and so $1/y_6=(9/77)-(172/9009)>1/63$, a contradiction.

(L) $\mathrm{L}({}_1d^z-{}_2d^z)=(\pm\chi(1), {}_1d^z-{}_2d^z)=(\pm\chi(1), {}_1d^z-d^t)=0$, $\mathrm{L}({}_1d^z-d^t)>0$ yield

(L1) $1+(9/y_1)+(9/y_2)+(1/y_6)-(16/y_7)-(16/y_8)=0$,

(L2) $1+y_1+y_2+y_6-y_7-y_8=0$,

(L3) $y_1+y_2+y_3=0$,

(L4) $(36/y_1)+(36/y_2)+(400/y_3)>0$.

$y_1\equiv 51$, $y_2\equiv 51$, $y_6\equiv 1$, $y_7\equiv 52$, $y_8\equiv 52 \pmod{64}$. By (L1), either y_1 or $y_2=-13$. So we may assume $y_1=-13$. Then (L4) implies $0<y_3<200$. If $y_3=154$, then (L3) gives $y_2=-141$, violating (*). Hence $y_3=90$, $y_2=-77$. If y_7 and y_8 both exceed 52, then (L1) and (L3) give $y_7=y_8=180$, $y_6=-63$, against (L2). So we may assume $y_7=52$. Then (L1) and (L2) imply $(1/y_6)-(16/y_8)=9/77$, $y_6-y_8=141$. Thus $-160<y_8<0$, so by (*) $y_8=-140$. Thus $y_6=1$, a contradiction.

(M) $\mathrm{L}({}_1d^z-{}_2d^z)=0$ yields

$$1-(1/x_1)-(1/x_2)-(1/x_3)+(16/x_4)+(1/y_2)+(1/y_3)-(16/y_4) = 0.$$

Also, $x_i\equiv 1$ $(1\leqq i\leqq 3)$, $x_4, y_4\equiv 52$, $y_2, y_3\equiv 1 \pmod{64}$. It follows easily that $x_4=-12$. Then $|(16/y_4)+(1/3)|\leqq 5/63$, so $-140<y_4<-12$; by (*) applied to a character of degree 12, $y_4\neq -76$, a contradiction.

Remark. These are the generalized decomposition numbers for $B_0(T\langle\beta\rangle)$.

(N) $\mathrm{L}(d^t)=\mathrm{L}({}_1d^z-{}_2d^z+d^t)=0$ yield

(N1) $1+(4/x_1)+(4/x_2)-(200/x_5)+(1/x_6)+(81/x_7)+(1/x_8)=0$,

(N2) $2+(3/x_1)+(3/x_2)+(16/x_3)+(16/x_4)-(200/x_5)+(2/x_6)+(2/x_8)=0$.

$x_1, x_2, x_6, x_8\equiv 1$; $x_3, x_4\equiv 52$; $x_5\equiv 26$, $x_7\equiv 41 \pmod{64}$; if $x_5>0$, then $x_5\leqq 90$, and if $x_7<0$, then $x_7\leqq -87$. First suppose $x_5=90$. (N1) implies $x_7=41$, which is impossible as $|x_i|\geqq 63$, $i=1, 2, 6, 8$. So $x_5\neq 90$. Then (N2) implies that we may assume $x_3=-12$. If also $x_4=-12$, then subtracting (N1) from (N2) we find $-81<x_7<0$, which is impossible. So $x_4\neq -12$. Now we can apply (*) to a character of degree 12. Thus $x_4\neq -76$. Suppose $x_5\neq 154$. By (*), $x_5>600$ if $x_5>0$. This contradicts (N2), so $x_5=154$. By (N2), $0<x_3<40$, a contradiction.

(P) $\mathrm{L}(d^t)=\mathrm{L}({}_1d^z-{}_2d^z)=0$ and $\mathrm{L}({}_1d^z-d^t)>0$ yield

(P1) $1+(4/x_1)+(4/x_2)-(200/x_5)+(1/x_6)+(25/x_7)+(25/x_8)=0$,

(P2) $1-(1/x_1)-(1/x_2)+(16/x_3)+(16/x_4)+(1/x_6)-(16/x_9)-(16/x_{10})=0$,

(P3) $(-4/x_1)-(4/x_2)+(64/x_3)+(64/x_4)+(400/x_5)>0$.

$x_1, x_2, x_6\equiv 1$, $x_3, x_4, x_9, x_{10}\equiv 52$, $x_5\equiv 26$, $x_7\equiv 53$, $x_8\equiv 53 \pmod{64}$; if $x_5>0$, then $x_5\geqq 90$. From (P1), we easily get $x_5>0$. If $x_5=90$, (P1) implies either x_7 or x_8 is >0 and <50, a contradiction. Hence $x_5\geqq 154$. If $x_4=-12$, then (P3) implies $0<x_3<32$, a contradiction. Therefore x_4, and similarly x_3, is $\neq -12$. Since $|x_i|\geqq 63$, $i=1, 2, 6$, it follows easily from (P2) that $x_3=x_4=-52$, $x_9=x_{10}=76$; thus $(1/x_1)+(1/x_2)-(1/x_6)=-9/247$. We may then assume that $x_1=-63$, and either $x_2 = -63$ or $x_6 = 65$. In either case the third x_i turns out not to be an integer, a contradiction.

(Q) $L(d^t + {}_1d^z - {}_2d^z) = 0$, $L({}_1d^z - d^t) > 0$ yield

(Q1) $2 + (3/x_1) + (16/x_2) + (16/x_3) + (16/x_4) - (75/x_5) + (2/x_7) - (16/x_8) = 0$,

(Q2) $(-4/x_1) + (64/x_2) + (64/x_3) + (64/x_4) + (100/x_5) > 0$.

$x_1, x_7 \equiv 1$, $x_2, x_3, x_4, x_8 \equiv 52$, $x_5 \equiv 37 \pmod{64}$; if $x_5 > 0$, then $x_5 \geqq 165$. Suppose $x_2 = -12$. Then (Q2) implies either x_3 or x_4 is positive and < 32, a contradiction. Hence $x_2 \neq -12$, and similarly for x_3 and x_4. Now (Q1) implies $75/x_5 > \frac{1}{2}$, so $0 < x_5 < 150$, a contradiction.

(R) $L(d^t) = (\pm\chi(1), d^t) = 0$ yield

(R1) $1 + (4/x_1) - (75/x_5) + (25/x_6) + (25/x_7) = 0$,

(R2) $1 + 4x_1 - 3x_5 + x_6 + x_7 = 0$.

$x_1 \equiv 1$, $x_5 \equiv 37$, $x_6 \equiv 53$, $x_7 \equiv 53 \pmod{64}$; if $x_5 > 0$, then $x_5 \geqq 165$. From (R2), it is impossible that $x_1, x_6, x_7 < 0$ and $x_5 > 0$ at the same time. Then (R1) easily yields $0 < x_5 < 225$, so $x_5 = 165$. Also from (R1), we may assume that $x_6 = -75$. Then we obtain $(4/x_1) + (25/x_7) = -7/33$, $4x_1 + x_7 = 569$. Therefore $x_7 \equiv 53 \pmod{256}$. From the first equation, $-165 < x_7 < 0$, a contradiction.

(S) From Lemma 5(b), $x_1 \equiv 51$, $x_4 \equiv 50$, $x_6, x_7 \equiv 53$, $x_8 \equiv 52 \pmod{64}$; if $x_4 > 0$, then $x_4 \geqq 114$. Now $0 > L(5d^t + 3{}_1d^z - 4{}_2d^z) \geqq 8 - (144/51) - (96/114) - (100/75) - (100/75) - (64/52)$, a contradiction.

(T) $L(d^t + {}_1d^z - {}_2d^z) = L(-d^t + 2{}_1d^z - 2{}_2d^z) = (\pm\chi(1), d^t + 3{}_1d^z - 4{}_2d^z) = 0$, $L({}_1d^z) > 0$ yield

(T1) $2 - (18/x_1) + (16/x_2) + (3/x_3) - (147/x_4) + (108/x_6) = 0$,

(T2) $1 + (72/x_1) + (32/x_2) - (6/x_3) - (243/x_5) = 0$,

(T3) $1 - 2x_4 - x_5 + x_6 = 0$,

(T4) $1 + (36/x_1) + (64/x_2) + (98/x_4) + (81/x_5) + (72/x_6) > 0$.

$x_1 \equiv 51$, $x_2 \equiv 52$, $x_3 \equiv 1$, $x_4 \equiv 39$, $x_6 \equiv 54 \pmod{64}$; if $x_5 < 0$, $x_5 \leqq -87$; if $x_6 < 0$, then $x_6 \leqq -138$. We show first that $x_2 \neq -12$, $x_1 \neq -13$. If $x_2 = -12$, (T2) implies $x_5 < 0$, and (T4) implies $98/x_4 > 2$, so $x_4 = 39$. As $x_5 < -87$, (T3) gives $x_6 < 0$. Then (T4) yields $98/x_4 > 3$, which is impossible. If $x_1 = -13$, (T2) implies $243/x_5 < -3$, so $-81 < x_5 < 0$, a contradiction. Now since $x_2 \neq -12$, (T1) gives $0 < x_4 < 295$. If $x_4 = 39$, (T1) implies $x_6 < 108$ and $x_6 > 0$, so $x_6 = 54$. Then (T3) implies $x_5 = -23$, a contradiction. If $x_4 = 167$, then (*) implies $|x_i| > 165$, $1 \leqq i \leqq 6$, and (T1) cannot hold, a contradiction. If $x_4 = 231$, (T1) implies $108/x_6 < -\frac{2}{3}$ so $x_6 = -138$; (T3) gives $x_5 = -599$, a prime, violating (*). Therefore, $x_4 = 103$. If $x_6 > 0$, (*) implies $|x_i| \geqq 102$, $1 \leqq i \leqq 3$, and (T1) cannot hold, a contradiction. Therefore $x_6 > 0$, and from (T3), $x_5 < 0$. By (*), $x_2 \leqq -140$ if $x_2 < 0$. Then (T2) implies $72/x_1 < -2/3$, so $-108 < x_1 < 0$, violating (*).

(U) We show in this case and (V) also that G has a rational character of degree 12, and prove (4.1). From Lemma 4,

(U1) $1 - (36/x_1) + (4/x_3) - (98/x_4) + (200/x_5) + (1/x_6) = 0$,

(U2) $1 + (18/x_1) + (16/x_2) - (1/x_3) - (49/x_4) + (1/x_6) - (16/x_7) = 0$, (U3) $1+(36/x_1)+(64/x_2)+(98/x_4)+(200/x_5)+(1/x_6)>0$.

Also, $x_1\equiv 51$, $x_2\equiv 52$, $x_3\equiv 1$, $x_4\equiv 39$, $x_5\equiv 42$, $x_6\equiv 1$, $x_7\equiv 52 \pmod{64}$; if $x_5<0$, then $x_5\leqq -150$. Suppose first that $x_2=-12$. Adding (U1) and (U3) yields $400/x_5>3$, so $x_5=42$ or 106. If $x_5=42$, then (U1) implies $(36/x_1)+(98/x_4)>5$, which is impossible. So $x_5=106$, violating (*). Thus, $x_2\neq -12$. Suppose $x_1\neq -13$. (U2) implies $0<x_4<228$. If $x_4=103$ or 167, then (U2) cannot hold without a violation of (*). Thus $x_4=39$. From (U1) we get $200/x_5>12/13$; thus $x_5=42$, 106, or 170. By (*) applied to a character of degree 39, $x_5\neq 106$. If $x_5=42$, then (U1) clearly cannot hold. So $x_5=170$. Now (U1) yields $-140<x_1<0$, so $x_1=-77$; again by (U1), $(4/x_3)+(1/x_6)<-1/10$, which is impossible. We have proved that $x_1=-13$. It now follows easily from (U1) that $x_4=39$, $x_5=-150$; then $(4/x_3)+(1/x_6)=1/13$, so $x_3=x_6=65$. From (U3), $64/x_2>7/12$, so $x_2=52$. (U2) yields $x_7=-12$. The character with degree $-x_7$ is clearly rational. By Lemma 4(c),

$$\frac{128|C_G(z)|^3}{|C_G(z,y)|^2} = |G|\left\{1-\frac{36}{13}+\frac{64}{52}+\frac{98}{39}-\frac{200}{150}+\frac{1}{65}\right\},$$

proving (4.1).

(V) We get the same equations as in (U), with $\sum_{i=1}^{4} 25/x_5^{(i)}$ substituted for $200/x_5$, and now $x_5^{(i)}\equiv 53 \pmod{64}$, $1\leqq i\leqq 4$. Suppose $x_1\neq -13$. If $x_2=-12$, then adding (U1) and (U3) we get $\sum_{i=1}^{4} 50/x_5^{(i)}>42/12$, so some $x_5^{(i)}=53$, violating (*). Thus if $x_1\neq -13$, then $x_2\neq -12$. As in (U), we conclude that $x_4=39$. Now (U1) implies that some $x_5^{(i)}$ is 53, again contradicting (*). Therefore $x_1=-13$. As in (U) we find $x_4=39$. Now (U1) gives $\sum_{i=1}^{4} 25/x_5^{(i)}<-964/819$. (*) applied to a character of degree 13 implies that $x_5^{(i)}<-200$ if $x_5^{(i)}<-75$. It follows that each $x_5^{(i)}=-75$. We can now argue as in (U).

5. **Completion of the proof.** Since G has a rational character of degree 12, a theorem of Schur [13] implies $|G| \mid 2^6\cdot 3^8\cdot 5^3\cdot 7^2\cdot 11\cdot 13$. We show $C_G(z)=C_G(Z(T))$. Let p be a prime divisor of $|O_{2'}(C_G(z))|$, and let P_0 and P be T-invariant Sylow p-subgroups of $O_{2'}(C_G(z))\cap C_G(Z(T))$ and $O_{2'}(C_G(z))$, respectively, with $P_0\leqq P$. Suppose $P_0<P$. Then from the character theory of T we conclude $p^4 \mid |P:P_0|$, so $p^4 \mid |C_G(z)|/|C_G(Z(T))|$. By (4.1), we get $p^{12} \mid |G|$, a contradiction. Therefore $P_0=P$ and, as p was arbitrary, $O_{2'}(C_G(z))\leqq C_G(Z(T))$. The structure of $C_G(z)$ modulo core yields $C_G(Z(T))=C_G(z)$. Let $N=N_G(Z(T))$. Thus N is strongly embedded in G. By a theorem of Bender [2], $G\cong Sz(8)$, $U_3(4)$, or $L_2(64)$, since $|T|=2^6$. As T has exactly 3 involutions, $G\cong U_3(4)$, completing the proof of Theorem 2.

We turn to the corollary to Theorem 1. Let N be a minimal normal subgroup of G. If $T\leqq N$, then $N=G$ and since T is indecomposable, G is simple; thus $G\cong U_3(4)$ by Theorem 1. So assume $T\nleqq N$.

If N is nonsolvable, then by the Z^*-theorem, N is simple and $N\geqq Z(T)$, since T contains only 3 involutions. As argued in the proof of Lemma 1, N contains an element α normalizing $N\cap T$ and cycling $Z(T)^\#$. Therefore $|N\cap T|\equiv 1 \pmod 3$. If $|N\cap T|=16$, then the existence of α implies that $N\cap T\cong Z_4\times Z_4$, contradicting the main theorem of [14]. So $N \cap T = Z(T)$, and $N \cong L_2(q)$ for some $q \equiv \pm 3 \pmod 8$, by [10]. But then $2^4 \nmid |\mathrm{Aut}\, N|$ so $C_G(N)$ contains an involution; this implies $C_G(N) \cap N \neq 1$, which is impossible.

Therefore N is solvable, so $N \leqq Z(T)$. If $|N| = 2$, then since $Z(T)$ is weakly closed in T, we get $Z(T)/N \lhd G/N$. Hence $Z(T) \lhd G$ in any case. Since $G = O^{2'}(G)$, $Z(T) \leqq Z(G)$. Denote residues modulo $Z(T)$ by bars. The proof of Lemma 1(c) implies that T has no automorphism of order 3 or 7 acting trivially on $Z(T)$. Hence 3 and 7 do not divide $|N_{\bar{G}}(\bar{T})/C_{\bar{G}}(\bar{T})|$. Clearly $\bar{G}$ is core free. By the main theorem of [14], a minimal normal subgroup of $\bar{G}$ is solvable, and it follows easily that $\bar{T} \lhd \bar{G}$. Therefore $T = G$, as required.

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